2. (a) Use integration by parts to find $\int x \sin 3x \, dx$.

(3)

(b) Using your answer to part (a), find $\int x^2 \cos 3x \, dx$.

(3)

a) $u = x \rightarrow \frac{du}{dx} = 1$

$v = -\frac{\cos 3x}{3} \leftarrow \frac{dv}{dx} = \sin 3x$

$I = -\frac{x\cos 3x}{3} - \int -\frac{\cos 3x}{3} \, dx$

$= -\frac{x\cos 3x}{3} + \int \frac{\cos 3x}{3} \, dx$

$= -\frac{x\cos 3x}{3} + \frac{\sin 3x}{9} + C$

b) $u = x^2 \rightarrow \frac{du}{dx} = 2x$

$v = \frac{\sin 3x}{3} \leftarrow \frac{dv}{dx} = \cos 3x$

$I = x^2\frac{\sin 3x}{3} - \int \frac{2}{3} x \sin 3x \, dx$ → already know $\int x \sin 3x \, dx$

$= \frac{x^2 \sin 3x}{3} - \frac{2}{3}\left[-\frac{x\cos 3x}{3} + \frac{\sin 3x}{9}\right]$

P 4 0 0 8 5 A 0 4 2 8

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Question 2 continued

$$= \frac{x^2 \sin 3x}{3} + \frac{2}{9} x \cos 3x - \frac{2}{27} \sin 3x + c$$

Q2

(Total 6 marks)

2. Using the substitution $u = \cos x + 1$, or otherwise, show that

$$\int_0^{\frac{\pi}{2}} e^{\cos x + 1} \sin x \;\; dx = e(e-1).$$

(6)

$$\frac{du}{dx} = -\sin x$$

$$dx = \frac{1}{-\sin x}\, du$$

Limits

x	$u = \cos x + 1$
0	2
$\frac{\pi}{2}$	1

$$\int_2^1 e^u \cancel{\sin x} \cdot \frac{1}{-\cancel{\sin x}}\, du$$

$$= \int_2^1 -e^u \, du$$

minus changes limits

$$= \int_1^2 e^u \, du$$

$$= \left[e^u \right]_1^2$$

$$= e^2 - e$$

$$= e(e-1)$$

8. (a) Find $\int (4y+3)^{-\frac{1}{2}}\,dy$

(2)

(b) Given that $y = 1.5$ at $x = -2$, solve the differential equation

$$\frac{dy}{dx} = \frac{\sqrt{(4y+3)}}{x^2}$$

giving your answer in the form $y = f(x)$.

(6)

a) $\dfrac{(4y+3)^{1/2}}{(4)(1/2)}$

$= \frac{1}{2}(4y+3)^{1/2}$

(b) $\int \frac{1}{\sqrt{4y+3}}\,dy = \int \frac{1}{x^2}\,dx$

$\Rightarrow \int (4y+3)^{-1/2} = \int x^{-2}\,dx$

from (a) $\Rightarrow$

$\Rightarrow \frac{1}{2}(4y+3)^{1/2} = -x^{-1} \quad (+c)$

At $y = 1.5$ $\quad \frac{1}{2}(4(1.5)+3)^{1/2} = -(-2)^{-1} + c$

$x = -2$

$c = 1$

Leave blank

Question 8 continued

$$\frac{1}{2}(4y+3)^{1/2} = -\frac{1}{x} + 1$$

$$(4y+3)^{1/2} = -\frac{2}{x} + 2$$

$$4y+3 = \left(\frac{-2}{x} + 2\right)^2$$

$$4y = \left(\frac{-2}{x} + 2\right)^2 - 3$$

$$y = \frac{1}{4}\left(\frac{-2}{x} + 2\right)^2 - \frac{3}{4}$$

P 3 8 1 6 0 A 0 2 3 2 4

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1. $$f(x) = \frac{1}{x(3x-1)^2} = \frac{A}{x} + \frac{B}{(3x-1)} + \frac{C}{(3x-1)^2}$$

(a) Find the values of the constants A, B and C.

(4)

(b) (i) Hence find $\int f(x)\,dx$.

(ii) Find $\int_1^2 f(x)\,dx$, leaving your answer in the form $a + \ln b$, where a and b are constants.

(6)

a) $$\frac{1}{x(3x-1)^2} = \frac{A}{x} + \frac{B}{(3x-1)} + \frac{C}{(3x-1)^2}$$

$$\frac{1}{x(3x-1)^2} = \frac{A(3x-1)^2 + B(3x-1) + Cx}{x(3x-1)^2}$$

so $\quad 1 \equiv A(3x-1)^2 + B(3x-1)x + Cx$

when $x = 0$ $\quad 1 \equiv A(-1)^2$

$\underline{A = 1}$

when $x = 1/3$ $\quad 1 \equiv 1/3\,C$

$\Rightarrow \underline{C = 3}$

when $x = 1$ $\quad 1 \equiv 4A + 2B + C$

$1 \equiv 4(1) + 2B + 3$

$2B = -6$

$\underline{B = -3}$

Leave blank

Question 1 continued

(b) $\int \frac{1}{x} + \frac{-3}{(3x-1)} + \frac{3}{(3x-1)^2} \, dx$

$= \int \frac{1}{x} - \frac{3}{(3x-1)} + 3(3x-1)^{-2} \, dx$

$= \ln x - \ln(3x-1) - (3x-1)^{-1} + c$

ii) $\left[\ln x - \ln(3x-1) - (3x-1)^{-1} \right]_1^2$

$= \left(\ln 2 - \ln 5 - \frac{1}{5}\right) - \left(\ln 1 - \ln 2 - \frac{1}{2}\right)$

$= \ln 2 - \ln 5 - \frac{1}{5} + \ln 2 + \frac{1}{2}$

$= 2\ln 2 - \ln 5 + \frac{3}{10}$

$= \ln 4 - \ln 5 + \frac{3}{10}$

$= \ln\left(\frac{4}{5}\right) + \frac{3}{10}$

Leave blank

7.

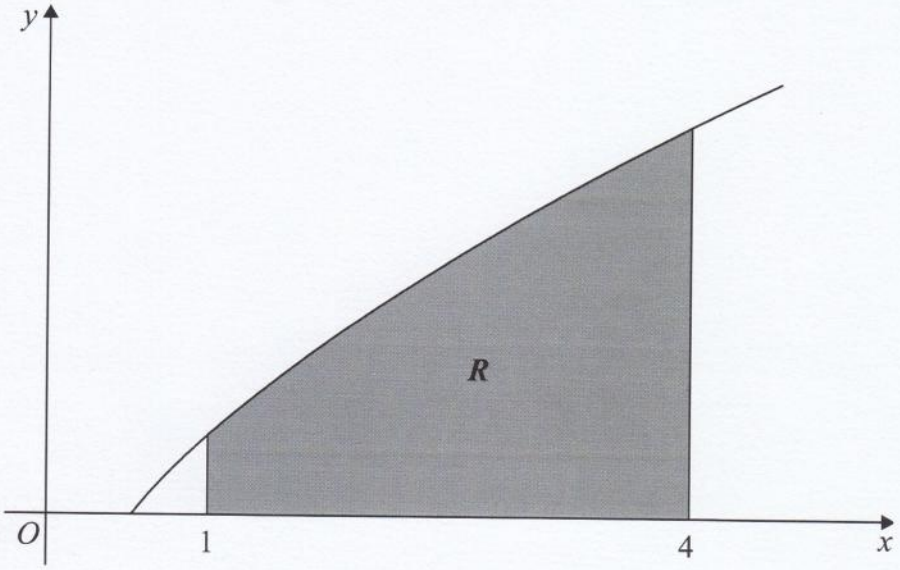

Figure 3

Figure 3 shows a sketch of part of the curve with equation $y = x^{\frac{1}{2}} \ln 2x$.

The finite region R, shown shaded in Figure 3, is bounded by the curve, the x-axis and the lines $x = 1$ and $x = 4$

(a) Use the trapezium rule, with 3 strips of equal width, to find an estimate for the area of R, giving your answer to 2 decimal places.

(4)

(b) Find $\int x^{\frac{1}{2}} \ln 2x \, dx$.

(4)

(c) Hence find the exact area of R, giving your answer in the form $a \ln 2 + b$, where a and b are exact constants.

(3)

a)

x	1	2	3	4
y	$\ln 2$	$\sqrt{2}\ln 4$	$\sqrt{3}\ln 6$	$2\ln 8$

Area $\approx \frac{1}{2}(1)\left[\ln 2 + 2\ln 8 + 2\left(\sqrt{2}\ln 4 + \sqrt{3}\ln 6\right)\right]$

Area ≈ 7.49

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Question 7 continued

(b) $\int x^{1/2} \ln 2x \, dx$ $u = \ln 2x \rightarrow \frac{du}{dx} = \frac{1}{x}$

$v = \frac{2}{3}x^{3/2} \leftarrow \frac{dv}{dx} = x^{1/2}$

$= uv - \int v \frac{du}{dx} \, dx$

$= \frac{2}{3}x^{3/2}(\ln 2x) - \int \frac{2}{3}x^{3/2} \cdot \frac{1}{x} \, dx$

$= \frac{2}{3}x^{3/2}(\ln 2x) - \int \frac{2}{3}x^{1/2} \, dx$

$= \frac{2}{3}x^{3/2}(\ln 2x) - \frac{4}{9}x^{3/2} + c$

(c)

$A = \int_1^4 x^{1/2} \ln 2x \, dx = \left[\frac{2}{3}x^{3/2} \ln 2x - \frac{4}{9}x^{3/2}\right]_1^4$

$= \left(\frac{16}{3}\ln 8 - \frac{32}{9}\right) - \left(\frac{2}{3}\ln 2 - \frac{4}{9}\right)$

$= \frac{16}{3}\ln 8 - \frac{32}{9} - \frac{2}{3}\ln 2 + \frac{4}{9}$

$= 16\ln 2 - \frac{2}{3}\ln 2 - \frac{28}{9}$

$= \frac{46}{3}\ln 2 - \frac{28}{9}$

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2. (a) Use integration to find

$$\int \frac{1}{x^3} \ln x \, dx$$

u

$\frac{dv}{dx}$

(5)

(b) Hence calculate

$$\int_1^2 \frac{1}{x^3} \ln x \, dx$$

(2)

a) $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$

$u = \ln x \qquad v = -\frac{1}{2}x^{-2}$

$\frac{du}{dx} = \frac{1}{x} \qquad \frac{dv}{dx} = x^{-3}$

$= -\frac{1}{2}x^{-2} \ln x$

$- \int -\frac{1}{2}x^{-2} \times x^{-1} \, dx$

$= -\frac{1}{2x^2} \ln x + \int \frac{1}{2x^3} dx$

$= -\frac{1}{2x^2} \ln x + \int \frac{1}{2} x^{-3} dx$

$= -\frac{1}{2x^2} \ln x - \frac{1}{2} \times \frac{1}{2} x^{-2} + c$

$= -\frac{1}{2x^2} \ln x - \frac{1}{4x^2} + c$

b) $\int_1^2 \frac{1}{x^3} \ln x \, dx = \left[\frac{-1}{2x^2} \ln x - \frac{1}{4x^2} \right]_1^2$

$= \left(-\frac{1}{8}\ln 2 - \frac{1}{16}\right) - \left(0 - \frac{1}{4}\right)$

$= -\frac{1}{8}\ln 2 - \frac{1}{16} + \frac{1}{4}$

$= -\frac{1}{8}\ln 2 + \frac{3}{16}$

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4.

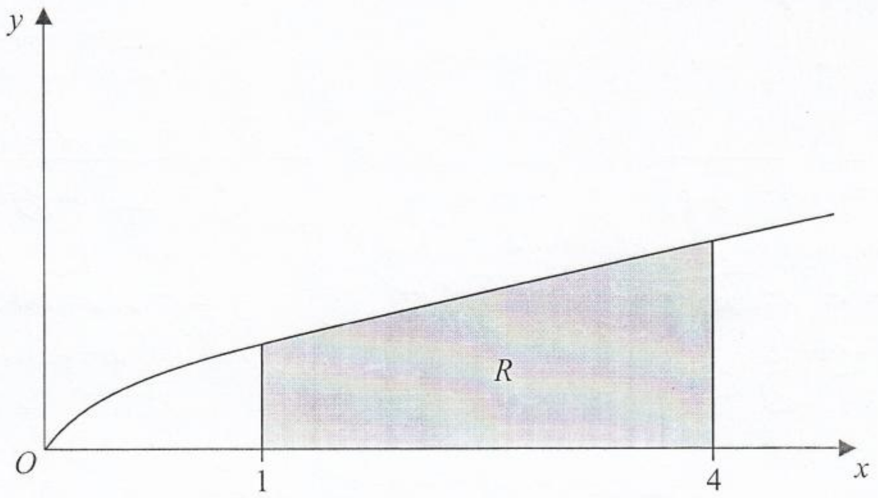

Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = \dfrac{x}{1+\sqrt{x}}$. The finite region R, shown shaded in Figure 1, is bounded by the curve, the x-axis, the line with equation $x = 1$ and the line with equation $x = 4$.

(a) Complete the table with the value of y corresponding to $x = 3$, giving your answer to 4 decimal places.

(1)

x	1	2	3	4
y	0.5	0.8284	1.0981	1.3333

(b) Use the trapezium rule, with all the values of y in the completed table, to obtain an estimate of the area of the region R, giving your answer to 3 decimal places.

(3)

(c) Use the substitution $u = 1 + \sqrt{x}$, to find, by integrating, the exact area of R.

(8)

b) Area $\simeq \frac{1}{2} \times 1 \times \left[0.5 + 1.3333 + 2(0.8284 + 1.0981)\right]$

≈ 2.84315

≈ 2.843 (3dp)

4c) $y = \dfrac{x}{1+\sqrt{x}}$

$$\int_1^4 \frac{x}{1+\sqrt{x}}\,dx$$

using substitution

$u = 1+\sqrt{x}$

$u = 1 + x^{\frac{1}{2}}$

$\dfrac{du}{dx} = \dfrac{1}{2}x^{-\frac{1}{2}}$

$\dfrac{du}{dx} = \dfrac{1}{2\sqrt{x}}$

$dx = 2\sqrt{x}\,du$

$u = 1+\sqrt{x}$

$u-1 = \sqrt{x}$

$(u-1)^2 = x$

Limits

$x=4,\ u = 1+\sqrt{4} = 3$

$x=1,\ u = 1+\sqrt{1} = 2$

$$\int_2^3 \frac{(u-1)^2}{u} \times 2 \times (u-1)\,du$$

$$= \int_2^3 \frac{(2u-2)(u^2-2u+1)}{u}\,du$$

$$= \int_2^3 \frac{2u^3-4u^2+2u-2u^2+4u-2}{u}\,du$$

$$= \int_2^3 \frac{2u^3-6u^2+6u-2}{u}\,du$$

$$= \int_2^3 2u^2-6u+6-\frac{2}{u}\,du$$

$$= 2\int_2^3 u^2-3u+3-\frac{1}{u}\,du$$

$$= 2\left[\frac{1}{3}u^3-\frac{3}{2}u^2+3u-\ln u\right]_2^3$$

$$= 2\left[\left(9-\frac{27}{2}+9-\ln 3\right)-\left(\frac{8}{3}-6+6-\ln 2\right)\right]$$

$$= 2\left[\left(\frac{9}{2}-\ln 3\right)-\left(\frac{8}{3}-\ln 2\right)\right]$$

$$= 2\left(\frac{11}{6}-\ln 3+\ln 2\right)$$

$$= 2\left(\frac{11}{6}+\ln\frac{2}{3}\right)$$

$$= \underline{\underline{\frac{11}{3}+2\ln\frac{2}{3}}}$$

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5.

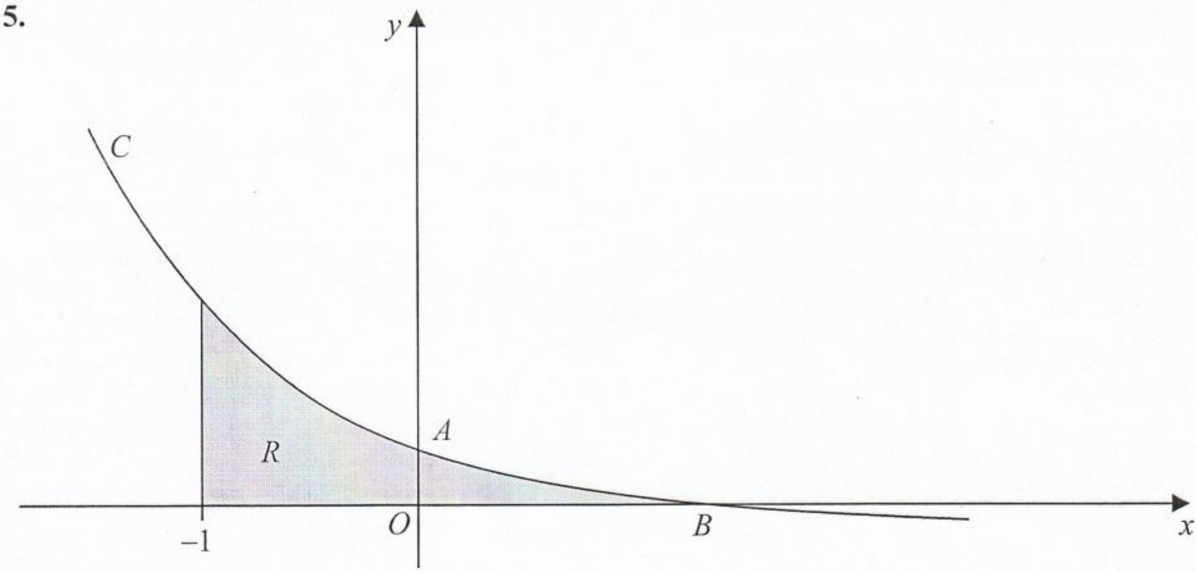

Figure 2

Figure 2 shows a sketch of part of the curve C with parametric equations

$$x = 1 - \frac{1}{2}t, \quad y = 2^t - 1$$

The curve crosses the y-axis at the point A and crosses the x-axis at the point B.

(a) Show that A has coordinates (0, 3).

(2)

(b) Find the x coordinate of the point B.

(2)

(c) Find an equation of the normal to C at the point A.

(5)

The region R, as shown shaded in Figure 2, is bounded by the curve C, the line $x = -1$ and the x-axis.

(d) Use integration to find the exact area of R.

(6)

a) y-axis, $x = 0$ $\quad 0 = 1 - \frac{1}{2}t$

$t = 2$

Sub $t=2$ $\quad y = 2^2 - 1 = 3$, so A is $(0,3)$

b) at B, $y = 0$, $0 = 2^t - 1$ so $t = 0$

sub $t = 0$, $x = 1 - \frac{1}{2} \times 0$, $x = 1$

x coordinate at B is $\underline{\underline{x = 1}}$

$x = 1 - \frac{1}{2}t$ $\qquad$ $y = 2^t - 1$

5c) $\frac{dx}{dt} = -\frac{1}{2}$ $\qquad$ $\frac{dy}{dt} = 2^t \ln 2$

learn this result

$\boxed{\begin{array}{l} \frac{d}{dt}(2^t) = 2^t \ln 2 \\ \frac{d}{dt}(3^t) = 3^t \ln 3 \end{array}}$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{2^t \ln 2}{-\frac{1}{2}} = -2^t \times 2 \ln 2$$

at A, $t = 2$

gradient at A, $\frac{dy}{dx} = (-2)^2 \times 2 \ln 2$

$$= 8 \ln 2$$

Gradient of normal at A is $\frac{-1}{8 \ln 2}$
(perpendicular)

$$y - y_1 = m(x - x_1)$$

$$y - 3 = -\frac{1}{8 \ln 2}(x - 0)$$

$\underline{y = -\frac{1}{8 \ln 2}x + 3}$ is equation of normal at C

5d) at $x = 1$, $1 = 1 - \frac{1}{2}t$, $t = 0$ (limits)
at $x = -1$, $-1 = 1 - \frac{1}{2}t$, $-2 = -\frac{1}{2}t$, $t = 4$

$$\int_{-1}^{1} y \, dx = \int_4^0 y \frac{dx}{dt} \, dt = \int_4^0 (2^t - 1) \times -\frac{1}{2} \, dt$$

$$= \frac{1}{2}\int_0^4 (2^t - 1) \, dt$$

switch limits getting rid of minus sign

$$= \frac{1}{2}\left[\frac{2^t}{\ln 2} - t\right]_0^4$$

$$= \frac{1}{2}\left[\left(\frac{16}{\ln 2} - 4\right) - \left(\frac{1}{\ln 2} - 0\right)\right]$$

$$= \frac{1}{2}\left(\frac{16}{\ln 2} - \frac{1}{\ln 2} - 4\right)$$

$$= \frac{1}{2}\left(\frac{15}{\ln 2} - 4\right)$$

$$= \underline{\frac{15}{2 \ln 2} - 2}$$

If $y = 2^t$

$\frac{dy}{dt} = 2^t \ln 2$

$\therefore$

$\int 2^t \, dt = \frac{2^t}{\ln 2} + c$

Leave blank

1. (a) Find $\int x^2 e^x \, dx$.

(5)

(b) Hence find the exact value of $\int_0^1 x^2 e^x \, dx$.

(2)

$$\int uv' = uv - \int u'v$$

$u = x^2 \quad v = e^x$

$u' = 2x \quad v' = e^x$

$$\int x^2 e^x \, dx = x^2 e^x - \int 2x e^x \, dx$$

$u \quad v'$

$u = x \quad v = e^x$

$u' = 1 \quad v' = e^x$

$$= x^2 e^x - 2\left[x e^x - \int e^x \, dx \right]$$

$$= x^2 e^x - 2\left(x e^x - e^x \right)$$

$$= x^2 e^x - 2x e^x + 2e^x$$

$$= \underline{\underline{e^x \left(x^2 - 2x + 2 \right) + C}}$$

b) $$\int_0^1 x^2 e^x \, dx = \left[e^x \left(x^2 - 2x + 2 \right) \right]_0^1$$

$$= \left(e(1 - 2 + 2) \right) - \left(e^0 (0 - 0 + 2) \right)$$

$$= \underline{\underline{e - 2}}$$

Leave blank

5. (a) Use the substitution $x = u^2$, $u > 0$, to show that

$$\int \frac{1}{x(2\sqrt{x} - 1)}\,dx = \int \frac{2}{u(2u - 1)}\,du$$

(3)

(b) Hence show that

$$\int_1^9 \frac{1}{x(2\sqrt{x} - 1)}\,dx = 2\ln\left(\frac{a}{b}\right)$$

where a and b are integers to be determined.

(7)

a) $x = u^2$

$\sqrt{x} = u$

$\frac{dx}{du} = 2u$

$dx = 2u\,du$

$\int \frac{1}{x(2\sqrt{x}-1)}\,dx = \int \frac{1}{\cancel{u^2}(2u-1)} \times 2\cancel{u}\,du$

$= \int \frac{2}{u(2u-1)}\,du$ as required

b) limits $x = 9$, $u = \sqrt{9} = 3$
$x = 1$, $u = \sqrt{1} = 1$

$\int_1^3 \frac{2}{u(2u-1)}\,du$ split into partial fractions

$\frac{2}{u(2u-1)} = \frac{A}{u} + \frac{B}{2u-1}$

$2 = A(2u-1) + Bu$

when $u = 0$, $2 = -A$ $\therefore A = -2$

when $u = \frac{1}{2}$, $2 = \frac{1}{2}B$ $\therefore B = 4$

$\int_1^3 -\frac{2}{u} + \frac{4}{2u-1}\,du$

P 4 3 1 3 7 A 0 1 6 3 2

5b (cont)

$$= \int_1^3 -\frac{2}{u}\,du + \int \frac{4}{2u-1}\,du$$

$$= \left[-2\ln u + \frac{4}{2}\ln(2u-1)\right]_1^3$$

$$= (-2\ln 3 + 2\ln 5) - (2\ln 1 + 2\ln 1)$$

$$= 2\ln 5 - 2\ln 3 - 0$$

$$< 2(\ln 5 - \ln 3)$$

$$= 2\ln\left(\frac{5}{3}\right)$$