Solving Differential Equations
Difficulty: Medium

Model Answers 2

Time allowed: **47 minutes**

Score: **/39**

Percentage: **/100**

Grade Boundaries:

A*	A	B	C	D	E	U
>76%	61%	52%	42%	33%	23%	<23%

Question 1

A population growth is modelled by the differential equation

$$\frac{dP}{dt} = kP\ ,$$

where P is the population, t is the time measured in days and k is a positive constant.

Given that the initial population is P_0,

(a) solve the differential equation, giving P in terms of P_0, k and t.

(4)

$\frac{dP}{dt} = kP$, *when* $t = 0, P = P_0$

Rearrange by separating the variables,

$$\int \frac{dP}{P} = \int k\ dt$$

$$\ln P = kt + c$$

Plug in the known values

$$\ln P_0 = k(0) + c$$

So we can say:

$$\ln P = kt + \ln P_0$$

Rearrange this for P using log rules

$$e^{\ln P} = e^{kt + \ln P_0} = e^{kt}e^{\ln P_0}$$

Hence $\boldsymbol{P = P_0 e^{kt}}$

Given also that $k = 2.5$,

(b) find the time taken, to the nearest minute, for the population to reach $2P_0$.

(3)

Here we plug in the given values and rearrange for t

$$(2P_0) = P_0 e^{2.5t}$$

$$2 = e^{2.5t}$$

$$ln\, 2 = ln(e^{2.5t}) = 2.5t$$

Hence

$$\boldsymbol{t = \frac{1}{2.5} ln\, 2 = 0.277\ days}$$

In minutes this is $\boldsymbol{0.277 \times 24 \times 60 = 399\ minutes}$

In an improved model the differential equation is given as

$$\frac{dP}{dt} = \lambda P \cos \lambda t,$$

where P is the population, t is the time measured in days and λ is a positive constant.

Given, again, that the initial population is P_0 and that time is measured in days,

(c) solve the second differential equation, giving P in terms of P_0, λ and t.

(4)

Similar to part a we must rearrange this and then integrate.

$$\int \frac{dP}{P} = \lambda \int dt \cos \lambda t$$

$$\ln P = \frac{\lambda}{\lambda} \sin \lambda t + c = \sin \lambda t + c$$

Plug in the known values→

$$\ln P_0 = \sin(0) + c$$

Hence

$$\ln P = \sin \lambda t + \ln P_0$$

Rearrange for P using exponentials

$$e^{\ln P} = e^{\sin \lambda t + \ln P_0} = e^{\sin \lambda t} e^{\ln P_0}$$

Therefore $\boldsymbol{P = P_0 e^{\sin \lambda t}}$

Given also that $\lambda = 2.5$,

(d) find the time taken, to the nearest minute, for the population to reach $2P_0$ for the first time, using the improved model.

(3)

We now do the same as before with the improved model.

$$P = 2P_0, \lambda = 2.5$$

$$2P_0 = P_0 e^{\sin 2.5t}$$

$$e^{\sin 2.5t} = 2$$

Use logarithms to rearrange for t→

$$\ln e^{\sin 2.5t} = \ln 2$$

$$\sin 2.5t = \ln 2$$

$$t = \frac{1}{2.5}\sin^{-1}(\ln 2) = 0.306 \ldots \textit{days}$$

$$t = 0.306 \ldots \times 24 \times 60 = \mathbf{441} \textit{ minutes.}$$

(Total 14 marks)

Question 2

(a) Express $\frac{5}{(x-1)(3x+2)}$ in partial fractions. **(3)**

$$let\ \frac{5}{(x-1)(3x+2)} = \frac{A}{x-1} + \frac{B}{3x+2}$$

$= \frac{A(3x+2)+B(x-1)}{(x-1)(3x+2)}$

$\Rightarrow \mathrm{A}(3x+2) + \mathrm{B}(x-1) = 5$

Using elimination:

x = 1 gives:

$\mathrm{A}(3+2) + \mathrm{B}(1-1) = 5$

5A = 5

A = 1

x = -3/2 gives:

$1\left(-\frac{9}{2}+2\right) + \mathrm{B}\left(-\frac{3}{2}-1\right) = 5$

$$-\frac{5}{2} - \frac{5}{2}\mathrm{B} = 5$$

$$-\frac{5}{2}\mathrm{B} = \frac{15}{2}$$

$$\mathbf{B = -3}$$

$\Rightarrow \frac{5}{(x-1)(3x+2)} = \frac{1}{x-1} - \frac{3}{3x+2}$

(b) Hence find $\int \frac{5}{(x-1)(3x+2)} \mathrm{d}x$, where $x > 1$. **(3)**

$\int \frac{5}{(x-1)(3x+2)} dx = \int \frac{1}{x-1} dx - \int \frac{3}{3x+2} dx =$. Using part a)

$= \boldsymbol{ln|x-1| - ln|3x+2| + C}$ where C is constant of integration

(c) Find the particular solution of the differential equation

$$(x-1)(3x+2)\frac{\mathrm{d}y}{\mathrm{d}x} = 5y, \quad x > 1,$$

for which $y = 8$ at $x = 2$. Give your answer in the form $y = \mathrm{f}(x)$. **(6)**

$$(x-1)(3x+2)\frac{dy}{dx} = 5y$$

Using the techniques of separation of variables, dividing LHS by y and RHS by (x-1)(3x+2)

$\int \frac{1}{y} \mathrm{dy} = \int \frac{5}{(x-1)(3x+2)} \mathrm{dx}$

Integrating on each side gives

$\ln|y| = \ln\left|\frac{x-1}{3x+2}\right| + \ln(k)$ where k is constant of integration

$y = \frac{k(x-1)}{3x+2}$

$8 = \frac{k}{8}$ using (2,8)

$\boldsymbol{y = \frac{64(x-1)}{3x+2}}$ **(Total 12 marks)**

Question 3

(a) Express $\frac{2}{P(P-2)}$ in partial fractions.

A team of biologists is studying a population of a particular species of animal.

The population is modelled by the differential equation

$$\frac{dP}{dt} = \frac{1}{2}P(P-2)\cos 2t, \; t \geq 0$$

where P is the population in thousands, and t is the time measured in years since the start of the study.

Given that $P = 3$ when $t = 0$,

(3)

$$\frac{2}{P(P-2)}$$

Express as a sum of fractions over the denominators

$$\frac{2}{P(P-2)} = \frac{A}{P} + \frac{B}{P-2}$$

Multiply both sides by $P(P-2)$

$$2 = \frac{A(P)(P-2)}{P} + \frac{B(P)(P-2)}{P-2}$$
$$2 \equiv A(P-2) + B(P)$$

Set $P = 0$ to eliminate B

$$2 \equiv A((0)-2) + B(0)$$

$$2 \equiv -2A$$

Divide both sides by -2

$$\boldsymbol{A = -1}$$

Set $P = 2$ to eliminate A

$$2 \equiv A\big((2) - 2\big) + B(2)$$

$$2 \equiv 2B$$

Divide both sides by 2

$$\boldsymbol{B = 1}$$

Substitute these back into the original equation

$$\frac{2}{P(P-2)} = \frac{A}{P} + \frac{B}{P-2}$$

$$\boldsymbol{\frac{2}{P(P-2)} = -\frac{1}{P} + \frac{1}{P-2} = \frac{1}{P-2} - \frac{1}{P}}$$

(b) solve this differential equation to show that

$$P = \frac{6}{3 - e^{\frac{1}{2}\sin 2t}} \qquad \textbf{(7)}$$

$$dP/_{dt} = \frac{1}{2}P(P-2)\cos 2t$$

Separate the variables and integrate both sides for

$$\int \frac{2}{P(P-2)}\,dP = \int \cos 2t\ dt$$

Substitute $\frac{2}{P(P-2)} = \frac{1}{P-2} - \frac{1}{P}$, from a), into the first integral

$$\int \frac{1}{P-2} - \frac{1}{P}\ dP = \int \cos 2t\ dt$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b|$$

$$\int \cos ax\ dx = \frac{1}{a}\sin ax$$

$$dP/_{dt} = \frac{1}{2}P(P-2)\cos 2t$$

Separate the variables and integrate both sides for

$$\int \frac{2}{P(P-2)}\ dP = \int \cos 2t\ \ dt$$

Substitute $\frac{2}{P(P-2)} = \frac{1}{P-2} - \frac{1}{P}$, from a), into the first integral

$$\int \frac{1}{P-2} - \frac{1}{P}\ \ dP = \int \cos 2t\ \ dt$$

$$\int \frac{1}{ax+b} dx = \frac{1}{a} ln|ax+b|$$

$$\int \cos ax\ \ dx = \frac{1}{a}\sin ax$$

remembering the $+C$

$$\ln|P-2| - \ln|P| = \frac{1}{2}\sin 2t + C$$

(c) find the time taken for the population to reach 4000 for the first time.
Give your answer in years to 3 significant figures. **(3)**

$P = 4$

Substitute this into $\ln\left|\frac{3\times(P-2)}{P}\right| = \frac{1}{2}\sin 2t$, found in the previous part, as it is the easiest equation to find t from

$$\ln\left|\frac{3\times((4)-2)}{(4)}\right| = \frac{1}{2}\sin 2t$$

$$\mathbf{\ln\left|\frac{6}{4}\right| = \frac{1}{2}\sin 2t}$$

Multiply both sides by 2

$$\sin 2t = 2\ln\frac{6}{4}$$

$\frac{6}{4} = \frac{3}{2}$, and do $\sin^{-1}$ on both sides

$$2t = \sin^{-1}\left(2\ln\frac{3}{2}\right)$$

Divide both sides by 2

$$t = \frac{1}{2}\sin^{-1}\left(2\ln\frac{3}{2}\right)$$

Calculate the answer, making sure the calculator is in *radians*

$$\mathbf{t = 0.47287\ldots}$$

(Total 13 marks)